Lower Bounds for Maximum Weight Bisections

of Graphs with Bounded Degrees

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Outline

1. Graphs and Weighted Graphs
2. Cuts, Bisections and colorings
3. Bounds for Maximum weight bisection
   - of graphs with even maximum degrees
   - of graphs with maximum degree three
4. triangle-free subcubic graphs
   - 2-connected
   - all cases except a claw

Definitions

Definition (Graphs)

A graph G is usually denoted by an ordered pair $G = (V(G), E(G))$, where $V(G)$ is the vertex sets and $E(G)$ is the edge set.

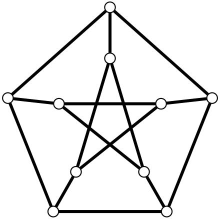

Figure: The Petersen graph

Definition (Weighted Graphs)

A weighted graph G is usually denoted by an ordered triple $G = (V(G), E(G), w)$ where $w : E \to \mathbb{R}_{\geq 0}$ is the weight function. In addition, we usually use $w(G) = \sum_{e \in E(G)} w(e)$ to denote the total weight of G.

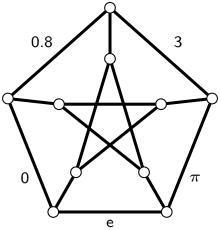

Figure: A weighted Petersen graph

Definition (neighbours and degrees)

Let G be a graph and v a vertex in $V(G)$. The neighbours of v are those vertices which are adjacent to v, and the degree of v, denoted by $d_G(v)$, is the number of neighbours of v.

Definition (the maximum degree)

Let G be a graph. The maximum degree of G denoted by $\Delta(G)$ is the maximum degree over all vertices, i.e., $\Delta(G) = \max\{d_G(v) : v \in V(G)\}$.

Definition (cubic and subcubic)

A graph is subcubic iff $\Delta(G) \leq 3$, and cubic iff the degree of any vertex equal three.

Cuts, Bisections and their values (or weights)

Definition

A cut of a weighted graph $G = (V, E, w)$ is a partition of the vertex set V into two disjoint subsets X and Y. The weight of the cut, denoted by $w(X, Y)$, refers to the sum of the weights over all edges between (i.e. edges with one endpoint in each partite set).

Definition

A bisection of G is a cut (X, Y) where the cardinality of X and Y differs by at most one.

The following is a well known result in graph theory by Erdős.

Theorem ([P. Erdős, 1965])

Every weighted graph $G = (V, E, w)$ has a cut with weight at least $\frac{w(G)}{2}$.

Proof.

State on the white board. □

Every graph can be seen as a weighted complete graph.

Lemma

Let $G = (V, E, w)$ be a weighted complete graph with n vertices. Then, G has a bisection with weight at least $\frac{n}{2(n-1)}w(G)$ when n is even and $\frac{n+1}{2n}w(G)$ when n is odd.

Proof.

State on the white board. □

Vertex Coloring

Definition (Vertex coloring)

A k-vertex-coloring of a graph G is a partition of the vertex set into k disjoint subsets such that there is no edge between any two vertices with the same color. A graph is said to be k-vertex-colorable if G has a k-vertex-coloring. The chromatic number $\chi(G)$ of a graph G refers to smallest positive integer k such that G is k-vertex-colorable.

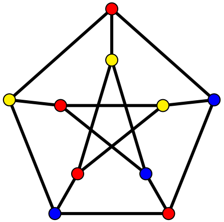

Figure: The Petersen graph

Theorem ([L. Anderson, D. Grant and N. Linial, 1983, J. Lehel and Zs. Tuza, 1982, S. Locke, 1982])

Let $G = (V(G), E(G), w)$ be a weighted graph and $\chi = \chi(G)$. Then, G admits a cut of weight at least $\frac{\chi+1}{2\chi}w(G)$ when χ is odd and $\frac{\chi}{2(\chi-1)}w(G)$ when χ is even.

Proof.

State on the white board. □

[1] $\chi(G) \leq \Delta(G) + 1$

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By Brook's theorem[1], we have the following holds.

Corollary

Let $G = (V(G), E(G), w)$ be a weighted graph. If $\Delta(G) \leq k$, then there exists a cut of weight at least $\frac{k+1}{2k}w(G)$ if k is odd and $\frac{k+2}{2(k+1)}w(G)$ if k is even.

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Our conjecture and past results

Conjecture ([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])

Let $G = (V(G), E(G), w)$ be a weighted graph. If $\Delta(G) \leq k$, then there exists a bisection of weight at least $\frac{k+1}{2k} w(G)$ if k is odd and $\frac{k+2}{2(k+1)} w(G)$ if k is even.

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Theorem ([B. Bollobás and A. Scott, 2004])

Let $G = (V(G), E(G))$ be a graph. If G is k-regular, then there exists a bisection of size at least $\frac{k+1}{2k}|E(G)|$.

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Theorem ([C. Lee, P. Loh and B. Sudakov, 2013])

Let $G = (V(G), E(G))$ be a graph. If $\Delta(G) \leq k$, then there exists a bisection of size at least $\frac{k+1}{2k}|E(G)| - \frac{k(k+1)}{4}$ if k is odd and $\frac{k+2}{2(k+1)}|E(G)| - \frac{k(k+2)}{4}$ if k is even.

Definition (Edge coloring)

A k-edge-coloring of a graph G is a partition of the edge set into k disjoint matchings. A graph is said to be k-edge-colorable if G has a k-edge-coloring. The chromatic index $\chi'(G)$ of a graph G refers to smallest positive integer k such that G is k-edge-colorable.

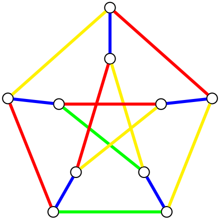


Figure: The Petersen graph

A Useful lemma

We say that $B \in \mathcal{B}_b(G)$, if B is the union of vertex-disjoint bipartite subgraphs B_i's (not necessarily connected) of G with partite sets (X_i, Y_i) where $G[X_i]$ and $G[Y_i]$ have no edges and $|X_i| = |Y_i|$.

Lemma

Let $G = (V, E, w)$ be a weighted graph and $B \in \mathcal{B}_b(G)$. Then, G has a bisection with weight at least $\frac{w(G)+w(B)}{2}$.

Proof.

State on the white board. □

Since any matching M of G is clearly in $\mathcal{B}_b(G)$, we have the following corollary.

Corollary

Let $G = (V, E, w)$ be a weighted graph and M its maximum weight matching. Then, G has a bisection with weight at least $\frac{w(G)+w(M)}{2}$.

Theorem

Every weighted graph $G = (V, E, w)$ has a bisection with weight at least $\frac{\chi'(G)+1}{2\chi'(G)} w(G)$.

Proof.

State on the white board. □

The following bound for chromatic index is known as Vizing's Theorem.

Theorem ([V. G. Vizing, 1964])

For any simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Corollary

Let k be a positive integer. Any weighted graph G with $\Delta(G) \leq k$ *has a bisection with weight at least*

$$\frac{k+2}{2(k+1)} w(G).$$

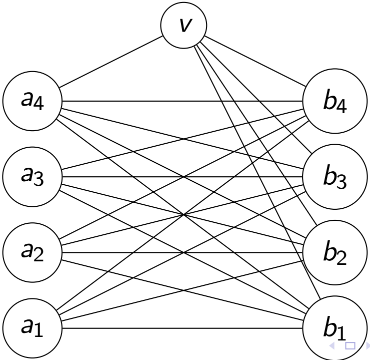
In particular, the conjecture holds when k is even.

Bounds for Maximum weight bisection

Why we cannot solve the odd case with edge coloring?

Conjecture ([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])

Let $G = (V(G), E(G), w)$ be a weighted graph. If $\Delta(G) \leq k$, then there exists a bisection of weight at least $\frac{k+1}{2k}w(G)$ if k is odd and $\frac{k+2}{2(k+1)}w(G)$ if k is even.



CONNECTED COMPONENTS, TREES AND FORESTS

1. A graph is connected if between any two vertices there is a path. Connected components of a graph are maximal connected subgraphs of it.
2. A tree is a connected graph with no cycle.
3. A forest is a graph of which all connected components are trees.

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Theorem

Trees are bipartite.

1. A graph is connected if between any two vertices there is a path. Connected components of a graph are maximal connected subgraphs of it.
2. A tree is a connected graph with no cycle.
3. A forest is a graph of which all connected components are trees.

Theorem

Trees are bipartite.

Theorem

Let F be a forest and $c(F)$ the number of connected components in F. Then,

$$|E(G)| = |V(G)| - c(F).$$

Lemma

If F is a forest with at most $|V(F)|/2$ edges, then there is a biparite subgraph $B \in \mathcal{B}_b(F)$ such that $E(B) = E(F)$.

Proof.

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Proof.

Let $c(F)$ be the number of connected components in F, m_i the number of components with i edges and t the maximum size of a component in F. Since F is a forest $|E(F)| = |V(F)| - c(F) \leq |V(F)|/2$, which implies that $|V(F)|/2 \leq c(F) = \sum_{i=0}^{t} m_i$. Thus,

$$\sum_{i=1}^{t} i \cdot m_i = |E(F)| \leq |V(F)|/2 \leq \sum_{i=0}^{t} m_i,$$

which implies $\sum_{i=2}^{t} m_i(i-1) \leq m_0$, and therefore we have enough isolated vertices to use. □

Definition

For any integer $k > 0$, a *k-bisection* of a graph G is a bisection (X, Y) where every component in $G[X] \cup G[Y]$ is a tree with at most k vertices.
Mattiolo and Mazzuoccolo showed the following result.

Lemma ([D. Mattiolo and G. Mazzuoccolo, 2021])

Every cubic multigraph has a 3-bisection.

Lemma

Every weighted cubic multigraph G has a bisection (X, Y) such that the following holds.

(i) $G[X] \cup G[Y]$ *is a forest;*

(ii) $|(X, Y)|$ *attains the maximum value among all bisection that satisfy (I);*

(iii) $\Delta(G[X]) \leq 1$ *and* $|E(G[Y])| \leq |Y|/2$.

Proof.

State on the white board. □

Theorem ([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])

Every weighted subcubic graph G has a bisection with weight at least $\frac{2}{3}w(G)$.

Sketch of the Proof

We may assume that G has at most one vertex with degree not equal to 3 as we may add edges of weight 0 between any two vertices with degree less than 3. We consider three cases.

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2. G has one vertex with degree one, say $d_G(z) = 1$.

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1. G is cubic.
2. G has one vertex with degree one, say $d_G(z) = 1$.
3. If G has a vertex with degree two, say $d_G(z) = 2$.

Triangle-free subcubic graphs (2-connected)

Theorem
([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])

Every bridgeless triangle-free subcubic graph G has a bisection with weight at least $\theta \cdot w(G)$, where $\theta = \frac{613}{855} \approx 0.716959$.

TRIANGLE-FREE SUBCUBIC GRAPHS (ALL CASES EXCEPT A CLAW)

Theorem
([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])

Every triangle-free subcubic graph G, different than the claw, has a bisection with weight at least $\theta \cdot w(G)$.

A conjecture

Conjecture ([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])

Every triangle-free subcubic graph G, different than the claw, has a bisection with weight at least $\frac{11}{15} \cdot w(G)$.

Thank you for your attention!

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